THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5000 Analysis I 2015-2016
Suggested Solution to Quiz 2

1. (a) Let $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $N > \frac{6}{\epsilon}$, i.e. $\frac{6}{N} < \epsilon$.
Then, for all $n \geq N$,
$$\left| \frac{2n}{n+3} - 2 \right| = \frac{6}{n+3} < \frac{6}{n} \leq \frac{6}{N} < \epsilon.$$
Therefore, $\lim_{n\to\infty} \frac{2n}{n+3} = 2$.

(b) Let $\epsilon > 0$, take $\delta = \sqrt{\epsilon} > 0$. Then for $|x - 0| < \delta$,
if $x \in \mathbb{Q}$, then
$$|f(x) - f(0)| = x^2 < \delta^2 < \epsilon;$$
if $x \in \mathbb{R}\backslash\mathbb{Q}$, then
$$|f(x) - f(0)| = 0 < \epsilon.$$
Therefore, f is continuous at $x = 0$.

2. (a) A bounded sequence in $\mathbb{R}$ has a convergent subsequence.

(b) Recall the fact that if $\{a_n\}$ is a sequence in $A \subset \mathbb{R}$ such that $\lim_{n\to\infty} a_n = L$ and $a_n \neq L$ for all $n \in \mathbb{N}$, then L is a cluster point of A.
Let $x_n = \sin n$, then it is a bounded sequence. By Bolzano-Weierstrass theorem, there exists a convergent subsequence x_{n_r}. Let $\lim_{r\to\infty} x_{n_r} = L$. We claim that L is a cluster point of S by showing that there is an infinite subsequence of $\{x_{n_r}\}$ such that each of them does not equal to L.
Suppose the contrary and so there are only finitely many terms of $\{x_{n_r}\}$ that do not equal to L. Then there exists a $K > 0$ such that $x_{n_r} = L$ for all $r \geq K$. Then
$$\begin{aligned} x_{n_{(k+1)}} &= x_{n_K} \\ \sin n_{(K+1)} &= \sin_{n_K} \\ n_{(K+1)} &= p\pi + (-1)^p n_K \end{aligned}$$
where p is an integer. Since $n_{(K+1)}$ and n_K are rational, p can only be zero. Then $n_{(K+1)} = n_K$ which leads a contradiction.

3. (a) Let $\epsilon > 0$. Since $\lim_{n\to\infty} x_n = L$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - L| < \epsilon$.
On the other hand,
$|x_n| \leq |x_n - L| + |L|$ implies that $|x_n| - |L| \leq |x_n - L|$ and
$|L| \leq |L - x_n| + |x_n|$ implies that $-|x_n - L| \leq |x_n| - |L|$.
They combine together and give $||x_n| - |L|| \leq |x_n - L|$.
Therefore, for all $n \geq N$, $||x_n| - |L|| \leq |x_n - L| < \epsilon$ and so $\lim_{n\to\infty} |x_n| = |L|$.

(b) Let $x_n = (-1)^n$ for $n \in \mathbb{N}$. Then $\lim_{n\to\infty} |x_n| = 1$ but $\{x_n\}$ does not converge.

4. (a) Suppose the contrary and so f is not bounded above.

   Then there exists a sequence $\{x_n\}$ in K such that $x_n \geq n$ for all $n \in \mathbb{N}$.

   By compactness of K, there is a subsequence $\{x_{n_r}\}$ in K such that $\lim_{r\to\infty} x_{n_r} = c \in K$.

   By continuity of f, we have $f(c) = f(\lim_{r\to\infty} x_{n_r}) = \lim_{r\to\infty} f(x_{n_r})$ which diverges to positive infinity. Therefore, it is a contradiction.

   (b) By (a), $f(K)$ is bounded above, therefore $\sup f(K)$ exists.

   Then for all $n \in \mathbb{N}$, there exists $x_n \in K$ such that

$$\sup f(K) - \frac{1}{n} < f(x_n) \leq \sup f(K).$$

   By compactness of K, there is a subsequence $\{x_{n_r}\}$ in K such that $\lim_{r\to\infty} x_{n_r} = x_M \in K$. Then,

$$\sup f(K) - \frac{1}{n_r} < f(x_{n_r}) \leq \sup f(K).$$

   By the sandwich theorem, $\lim_{r\to\infty} f(x_{n_r}) = \sup f(K)$ and by the continuity of f, $f(x_M) = \sup f(K)$.